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On Time-Dependent Transformation in Newtonian Mechanics: Formal Theory on the Relativity of Force

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Abstract

F. Klein's viewpoint of geometry is applied to the discussion of the time-dependent transformation (TDT) in Newtonian mechanics. Formally the vectorial character of transformation and the invariance of the form of Newtonian equations of motion under TDT are assumed. Then it is shown that our conception of force in Newtonian equations satisfies a certain transformation property under TDT, which is different from the case of Galileo and Newton in which force is regarded as an absolute quantity. Our theory also enables one to derive the fictitious forces such as the centrifugal force and Coriolis force. The balance equation between these fictitious forces and those with reversed sign is considered. The centripetal force of the new type and the force depending on angular acceleration are also given.

1. *Introduction*

It is well known that, from the geometrical viewpoint, Newtonian mechanics belongs to the category of Euclidean geometry. According to F. Klein, Euclidean geometry is "the science whose purpose is to study the invariant properties of figures under any transformation belonging to the group of motions" (Yano, 1968). In other words, we can state the following: "Euclidean geometry is the geometry that has the group of motions as its fundamental group." Here in three-dimensional space $x \equiv {}^t(x_1, x_2, x_3)$, the element T_3' of the group of motions can be written as follows:

$$T_3' : x \to x' = A \cdot x + a \tag{1.1}$$

Here we use the symbols, $x' \equiv {}^t(x_1', x_2', x_3')$, $a \equiv {}^t(a_1, a_2, a_3)$, and $A \equiv (a_{ij})$, and we have the following requirements:

$$\det|a_{ij}| = 1 \quad \text{and} \quad {}^tA = A^{-1} \tag{1.2}$$

All real numbers a_{ij} and a_k are constants independent of x. As is well known, the set of these transformations T_3' forms a group. The group T_3' corresponding to the case $a = 0$ is called the rotational group, and we write this as T_3. The group T_3' corresponding to the case $A = 1$ is called the translational group.

On the other hand, in Newtonian mechanics, when we choose *arbitrarily* a single observer, there exists some degree of freedom, or arbitrariness, in his choice of a coordinate system (including time t) $(x, t) \equiv {}^t(x_1, x_2, x_3, t)$ which he uses to describe the motion of a point-particle. (Ordinarily, in a textbook the condition of "inertial frame" is imposed on this coordinate system. However, we neglect this condition for a while.) We express this arbitrariness in terms of a transformation T. Let the other coordinate system, which the same observer can use, be (x', t'). Then we have the expression

$$T\colon (x, t) \to (x', t') \tag{1.3}$$

Here the transformation T consists of two transformations T_3' and T_0; the transformation $x \to x'$ is the element of the group T_3' of motions and the transformation $t \to t'$ is that of the translational group T_0 ($t \to t' = t + a_0$, a_0 is an arbitrary real constant). Thus the transformation T consists of two mutually independent transformations T_3' and T_0, and therefore T forms a group. For these transformations T, two differentials $dx \equiv {}^t(dx_1, dx_2, dx_3)$ and dt suffer "rotation" T_3 and "identical transformation" 1:

$$T_3\colon dx \to dx' = A \cdot dx, \qquad 1\colon dt \to dt' = 1 \cdot dt = dt \tag{1.4}$$

Therefore the set of ordered three differentials ${}^t(dx_1, dx_2, dx_3)$ can be considered as a vector in three-dimensional Euclidean space E^3, whose length squared is defined by $(dx_1)^2 + (dx_2)^2 + (dx_3)^2$. The differential dt is an invariant of the transformation T. Therefore we need only consider the former in equations (1.4); therefore we need only consider the transformation T_3' in equation (1.1) as the transformation of coordinates x. (In this discussion it is necessary that the transformation T does not depend on x or t; i.e., T is a "constant transformation." This point will be discussed later.) Therefore if we define the velocity $v \equiv {}^t(v_1, v_2, v_3)$ of a point particle by $dx_i = v_i dt$ ($v_i = dx_i/dt$), then v is a vector in E^3. In the same way, for the acceleration $\alpha_i = dv_i/dt = d^2x_i/dt^2$, $\alpha \equiv {}^t(\alpha_1, \alpha_2, \alpha_3)$ is a three-vector. Let mass m be a 3-scalar and force $F \equiv {}^t(F_1, F_2, F_3)$ be a 3-vector. Then Newton's equation of motion,

$$m\frac{d^2x_i}{dt^2} = F_i, \qquad i = 1, 2, 3 \tag{1.5}$$

is a vector equation (therefore a tensor equation) and it is a covariant (i.e., form-invariant) relation for the transformation T_3' and therefore it has a geometrical meaning. For the transformation T_3' of equation (1.1), we have the following transformation properties:

$$\frac{d^2x}{dt^2} \to A \cdot \frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2}$$

$$F \to A \cdot F = F' \tag{1.6}$$

and

$$m \to m = m'$$

Therefore equation (1.5) is transformed, under the same T_3', as

$$m\frac{d^2x_i'}{dt^2} = F_i', \qquad i = 1, 2, 3 \tag{1.5'}$$

and its form remains invariant. (*The covariance* of the equation of motion automatically follows because of its *vectorial character.* In the next section we shall require formally these two properties.) Accordingly when we choose a single observer arbitrarily, equation (1.5) does not depend on the choice of an orthogonal coordinate system (with positive orientation) used by the observer. Furthermore, because the choice of the observer is arbitrary as previously stated, equation (1.5) is meaningful for all observers. In Newtonian mechanics the differential of time dt is common, i.e., invariant ($dt' = dt$) for the transformation $(x, t) \to (x', t')$ between two observers, and therefore we can verify definitely the above fact. (In the following sections we assume further that time t is an absolute parameter: $t' = t$.) In the following we consider the transformation to the accelerated frame requiring formally the vectorial transformation property and the form invariance of Newton's equation.

2. *Transformation between Observers: The Problem of the Accelerated Frame*

a. The Generalized Galileo-Transformation. First we consider the following transformation:

$$dv_i \to dv_i' = dv_i - \alpha_i\, dt, \qquad i = 1, 2, 3 \tag{2.1}$$

which is the generalization of the classical Galileo transformation and which corresponds to the "Galileo transformation in velocity space." Here v and v' are the velocities of a point particle in two reference frames that correspond to two observers S and S' and $\alpha = {}^t(\alpha_1, \alpha_2, \alpha_3)$ is the relative acceleration of S' with respect to S and it may generally be dependent on time; $\alpha = \alpha(t)$. This generalized Galileo transformation (2.1) corresponds, as the transformation of space coordinates, to the case $A = 1$ (unit matrix) and a depends on time t in equation (1.1). For this transformation equation (1.4) does not hold, and therefore equation (1.5) cannot be regarded as a vectorial equation. However, in this paper we proceed requiring formally the vectorial transformation-property of Newton's equation (1.5) under the transformation (2.1), as in

Section 1. This requirement holds as an approximation, locally in t or in the "moment," if the time dependence of $a(t)$ is small, i.e.,

$$\left|\frac{da(t)}{dt}\right| \ll \left|\frac{dx}{dt}\right| \text{ or } \left|\frac{dx'}{dt}\right| \quad \text{(whichever is larger)}$$

Then under this generalized Galileo transformation the left-hand side of Newton's equation is transformed as

$$m\frac{d^2x}{dt^2} \to m1 \cdot \frac{d^2x}{dt^2} = m\left(\frac{d^2x'}{dt^2} + \alpha\right)$$
$$= m\left(\frac{d^2x'}{dt^2} - \alpha'\right) \qquad (2.2)$$

setting formally $A = 1$ in equation (1.6). Here we used equation (2.1); $\alpha' \equiv {}^t(\alpha'_1, \alpha'_2, \alpha'_3)$ is the acceleration of the S frame with respect to the S' frame. If we require that the force F acting on the point particle m be transformed under this transformation, as

$$F \to 1 \cdot F = F \qquad (2.3)$$

according to (1.6), then Newton's equation is transformed as follows:

$$m\left(\frac{d^2x'}{dt^2} - \alpha'\right) = F \qquad (2.4)$$

On the other hand if we require the covariance of Newton's equation, we have the equation (1.5′) in the S' frame. Here $F' = {}^t(F'_1, F'_2, F'_3)$ represents the force acting on the *same* point particle m with respect to the S' frame. Therefore we have the following equation:

$$F = F' - m\alpha' \qquad (2.5)$$

This equation shows the relation existing between the force F for the observer S and the force F' for the observer S'. Here the relative acceleration α' can generally depend on time.

In classical Newtonian mechanics the relation $F' = F$ is presupposed (Møller, 1952), and therefore Newton's equation cannot be covariant under the generalized Galileo transformation. It is well known that the form invariance under the condition $F' = F$ holds only in the case where α' ($= -\alpha$) is zero. Ordinarily textbooks treat only this case as a Galileo transformation. However, in our viewpoint, if the forces satisfy equation (2.5), then Newton's equation can have its meaning for such an accelerated frame as $\alpha' \neq 0$. Our relation (2.5) shows that two concepts of force and (mass x acceleration) are equivalent, and the concept of force is not absolute if we take into account accelerated frames. In the next section we consider the transformation to the rotating reference frame and show that the centrifugal force and Coriolis force can be introduced in the same way as above.

b. The Rotating Frame. Let us consider the transformation from a fixed reference frame S to a rotating reference frame S'. Here the angular velocity of the rotating frame can depend on time. That is, Euler angles (Goldstein, 1950) ϕ, θ, and ψ need not be linear formulas of time t. Let the coordinates of a point particle be transformed from x to x' under this transformation. Then we have,

$$\begin{aligned} x' &= A(t) \cdot x, \qquad && A(t) \equiv (a_{ij}(t)) \\ \det|a_{ij}(t)| &= 1, && \text{for all } t \\ A^{-1}(t) &= {}^tA(t) \end{aligned} \tag{2.6}$$

Here the matrix $A(t)$ may depend on t. Equation (2.6) corresponds to the case where $a = 0$ and $A = A(t)$ in equation (1.1). Equation (1.4) does not hold either in this case, and therefore Newton's equation cannot have the vectorial transformation property under equation (2.6), as in the case in Section 2.a. (This point will be discussed later.) However, we require formally also here the following two things: the vectorial transformation property and the covariance of Newton's equation. Then under the transformation (2.6), the acceleration and the force transform as equation (1.6). On the other hand we have the following equations differentiating equation (2.6):

$$\frac{dx'}{dt} = \frac{dA}{dt} \cdot x + A \cdot \frac{dx}{dt}, \qquad \left(\frac{dA}{dt}\right)_{ij} \equiv \frac{da_{ij}(t)}{dt} \tag{2.7}$$

and

$$\frac{d^2x'}{dt^2} = \frac{d^2A}{dt^2} \cdot x + 2\frac{dA}{dt} \cdot \frac{dx}{dt} + A \cdot \frac{d^2x}{dt^2} \tag{2.8}$$

Therefore we have the equation

$$m\frac{d^2x}{dt^2} \to mA \cdot \frac{d^2x}{dt^2} = m\left\{\frac{d^2x'}{dt^2} - \frac{d^2A}{dt^2} \cdot x - 2\frac{dA}{dt} \cdot \frac{dx}{dt}\right\} = A \cdot F \tag{2.9}$$

Next we assume formally $dA/dt = 0$ in equation (2.7). [If the time dependence of $A(t)$ is much slower than that of $x(t)$ or $x'(t)$, then we can accept the assumption locally in t or in the "moment."] Thus we have

$$\frac{dx'}{dt} = A(t) \cdot \frac{dx}{dt} \tag{2.10}$$

and therefore velocity can be regarded as a vector (or a tensor). That is, in the "moment" velocity dx/dt has the tensorial character, if the temporal change of $A(t)$ is slow. Under this condition we have the equation

$$mA\frac{d^2x}{dt^2} = F' - m\frac{d^2A}{dt^2} \cdot A^{-1}x' - 2m\frac{dA}{dt} \cdot A^{-1}\frac{dx'}{dt} \tag{2.11}$$

where we used the covariance of Newton's equation (1.5′).

Thus we have

$$F \to A \cdot F = F' - m\,\frac{d^2A}{dt^2} \cdot A^{-1}x' - 2m\,\frac{dA}{dt} \cdot A^{-1}\frac{dx'}{dt} \tag{2.12}$$

The force $A \cdot F$ is nothing but the representation of the force F acting on the point particle in the S frame by the components along axes of the S' frame. In a special case, the second term of (2.12) represents the centrifugal force and the third term the Coriolis force. (On the direction of the force corresponding to the third term, see the remark given later.) This fact can be easily verified, if we take the expression

$$A(t) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \phi = \phi(t) \tag{2.13}$$

which represents the ϕ rotation around the x_3 axis. [See equation (3.6) below.]

In the preceding discussion we assumed the equation $dA/dt = 0$ in the formula of the transformation of velocity (2.7). Next we consider the case without this assumption.

3. *Nontensorial Character of Velocity: The Relativity of Coriolis Force*

Let the point particle be constrained to move on a spherical surface. In this case we can take $A(t)$ of equation (2.6) such that $dx'/dt = 0$. That is, there exists the transformation $A(t)$ which gives the new frame S' in which the velocity of a point particle becomes zero, and which satisfies the condition (2.6). If the transformation $x \to x' = A \cdot x$ gives the equation $dx'/dt = 0$ ($dx/dt \neq 0$), then Coriolis force can be eliminated in the S' frame. If in the general formula of velocity transformation (2.7) we put $dx'/dt = 0$, then we have

$$0 = \frac{dA}{dt} \cdot x + A \cdot \frac{dx}{dt} \tag{3.1}$$

and

$$\frac{dx}{dt} = -A^{-1}\,\frac{dA}{dt} \cdot x \tag{3.2}$$

The latter gives the velocity dx/dt of a particle at a point x when $A(t)$ is given. The former gives $A(t)$ when dx/dt is known. This is a simultaneous linear homogeneous differential equation of order 1. Therefore the three independent matrix elements of $A(t)$ are uniquely determined in terms of the "coefficients" $x(t)$ and $dx(t)/dt$, and the initial value of $A(t)$.[1] If we use such a transformation as $A(t)$, the Coriolis force becomes zero in the S' frame.

[1] The linearity of equation (3.1) may be destroyed when it is expressed by the independent elements of $A(t)$. In this case the above discussion still holds locally in t by the general theory of differential equations (e.g., Shimizu, 1965).

Using the general formula of velocity transformation (2.7) in equation (2.9), we obtain the following equation instead of equation (2.11):

$$mA\frac{d^2x}{dt^2} = F' - m\frac{d^2A}{dt^2}A^{-1}x' + 2m\left(\frac{dA}{dt}A^{-1}\right)^2 x' - 2m\frac{dA}{dt}A^{-1}\frac{dx'}{dt} \tag{3.3}$$

Thus we have the equation

$$F \rightarrow A \cdot F = F' - m\frac{d^2A}{dt^2}A^{-1}x' + 2m\left(\frac{dA}{dt}A^{-1}\right)^2 x' - 2m\frac{dA}{dt}A^{-1}\frac{dx'}{dt} \tag{3.4}$$

The third term in the right-hand side of this equation is the force introduced newly. Let the transformation of coordinates $x \rightarrow x'$ be given by $A(t)$ of equation (2.13). This corresponds to, as the transformation of coordinate axes, the rotation $\phi(t)$ of the S' frame around the x_3 axis of a fixed S frame. [This is expressed symbolically as $S' = A(t) \cdot S$.] If we observe the motion of the point-particle m, in the S frame, which moves straight with a constant velocity in the S' frame, i.e., $F' = 0$, then we have the equation according to equation (3.4),

$$A \cdot F = -m\frac{d^2A}{dt^2}A^{-1}x' + 2m\left(\frac{dA}{dt}A^{-1}\right)^2 x' - 2m\frac{dA}{dt}A^{-1}\frac{dx'}{dt} \tag{3.5}$$

The particle m does the motion corresponding to the force F given by equation (3.5). If we use the special expression (2.13) for A, we have the equation

$$A \cdot F = m\dot{\phi}^2\begin{pmatrix} x_1' \\ x_2' \\ 0 \end{pmatrix} + m\ddot{\phi}\begin{pmatrix} -x_2' \\ x_1' \\ 0 \end{pmatrix} - 2m\dot{\phi}^2\begin{pmatrix} x_1' \\ x_2' \\ 0 \end{pmatrix} + 2m\dot{\phi}\begin{pmatrix} -dx_2'/dt \\ dx_1'/dt \\ 0 \end{pmatrix} \tag{3.6}$$

The first two terms of this equation come from the first term of equation (3.5). The first of them is the centrifugal force, and the second is the deflecting force by the angular acceleration. The third term of equation (3.6) is the force introduced newly and is the centripetal force towards the origin of the coordinates. The magnitude of this centripetal force is just equal to twice the centrifugal force. The vectorial sum of them is the centripetal force whose magnitude is equal to that of the first centrifugal force. The fourth term is the reversed Coriolis force (Goldstein, 1950). Let us consider this motion for the case $\dot{\phi}$ = const. The locus of the particle in the S' frame is the straight line (with direction) starting from, e.g., the origin, and in the S frame it is the eddy outward from the origin. The eddy cuts across the positive x_1 axis with the constant "period" $2\pi/\dot{\phi}$, and we have $\overline{\beta_1\beta_2} = \overline{\beta_2\beta_3} = \cdots$, where β_i are the points at which the eddy cuts across the positive x_1 axis. If the third term, centripetal force, of equation (3.6) does not exist, then the particle in the S frame will be flung away from the origin by the centrifugal force more rapidly than the eddy having the above property. If the fourth term, the reversed Coriolis force, does not exist, then we will have $\overline{0\beta_1} < \overline{\beta_1\beta_2} < \overline{\beta_2\beta_3} < \cdots$ and the orbit of the particle will expand, so that

(the time between β_1 and β_2) < (the time between β_2 and β_3) < $\cdots$

In any case, if we observe, in a fixed frame, the motion of the particle that moves straight with a constant velocity in a rotating frame (we do not assume the constancy of $\dot{\phi}$ here), then we observe a motion as if the four forces of equation (3.6) were actually acting on the particle. Conversely, only in this circumstance does the motion of a particle in a rotating frame become straight motion with constant velocity. It is to be noted that $A \cdot F$ in equation (3.6) is the expression of the force F in the S frame using its components along the orthogonal axes of the S' frame.The force depending on the angular acceleration $\ddot{\phi}$ vanishes if the angular velocity $\dot{\phi}$ is constant. Ordinarily texts treat only this case. However, this term may become important when an external force (e.g., electromagnetic field, etc.) is applied to the system.

This discussion shows the physical reality of equation (3.4) including our newly introduced centripetal force, viz., the third term of equation (3.4).

4. *The Relativity between a Fixed Frame and a Rotating Frame, and Balance of Forces: The Symmetry of the Theory*

In the preceding sections we have formally distinguished between a fixed frame and a rotating frame (or an accelerated frame). However this distinction is physically meaningless, if we admit a negative angle of rotation (only in this case does the set of these rotational transformations form a group): It is essentially arbitrary (or relative) what we call a fixed frame (or an accelerated frame) between two observers S and S'. In this section we investigate the symmetry of our formulation on this relativity.

The following two descriptions represent a physically identical situation (the relativity between a fixed frame and a rotating frame):

(1) Observing from the S frame, the S' frame is rotating with an angle $\phi(t)$ around the x_3 axis (which, we assume, coincides with the x_3' axis).

(2) Observing from the S' frame, the S frame is rotating with an angle $-\phi(t)$ around the x_3' axis (which coincides with the x_3 axis).

In the case (1) the force F' ($= md^2x'/dt^2$) is given by equation (3.4) as

$$F' = A \cdot F + m\frac{d^2A}{dt^2}A^{-1}x' - 2m\left(\frac{dA}{dt}A^{-1}\right)^2 x' + 2m\frac{dA}{dt}A^{-1}\frac{dx'}{dt} \qquad (4.1)$$

Case (2) corresponds to the following formal interchanges in case (1):

$$x \leftrightarrow x', \qquad F \leftrightarrow F', \qquad A(\phi(t)) \leftrightarrow A(-\phi(t)) = A^{-1}(\phi(t)) \qquad (4.2)$$

Here we write $A(t)$ of equation (2.13) as $A(\phi(t))$. Actually we can easily verify the invariance of equation (4.1) under the interchange (4.2) using the relations

$$x' = A(\phi(t)) \cdot x$$

and

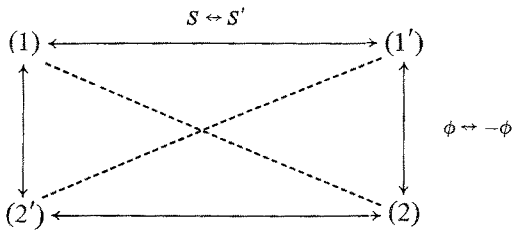
$$\frac{d(A^{-1})}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}$$
$$\frac{d^2(A^{-1})}{dt^2} = 2A^{-1}\left(\frac{dA}{dt}A^{-1}\right)^2 - A^{-1}\frac{d^2A}{dt^2}A^{-1} \tag{4.3}$$

Furthermore we introduce the following two descriptions, (1′) and (2′), which correspond to the formal interchange between S and S' in (1) and (2), respectively (this corresponds to the interchanges $x \leftrightarrow x'$ and $F \leftrightarrow F'$):

(1′) Observing from the S' frame, the S frame is rotating with an angle $\phi(t)$ around the x_3' axis.

(2′) Observing from the S frame, the S' frame is rotating with an angle $-\phi(t)$ around the x_3 axis.

The following "commutative diagram" holds among these four situations:



Here the horizontal arrow represents the interchange between the S frame and the S' frame (i.e., the two interchanges $x \leftrightarrow x'$ and $F \leftrightarrow F'$), and the vertical arrow represents the interchange between $\phi(t)$ and $-\phi(t)$. There exist two routes (1) → (1′) → (2) and (1) → (2′) → (2) in the diagram to go to (2) from (1); these two routes give the identical result (2) ("commutative"). In fact we can easily verify that when we operate the two interchanges $S \leftrightarrow S'$ and $\phi \leftrightarrow -\phi$, in equation (4.1), we have the identical result independent of their order. The dotted lines in the diagram show the physical equivalence stated above.

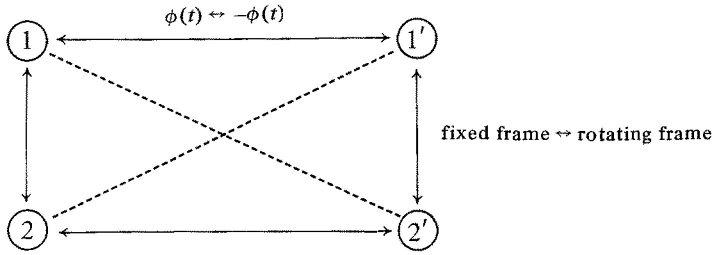
Let us consider the case in which an observer (S' frame) observes the motion of a particle that corresponds to the free motion ($F = 0$) to the other observer (S frame). Here we assume that the two frames of S and S' are rotating relatively around the x_3 axis (which is identical to the x_3' axis). There are the following four descriptions according to whether we call the fixed frame S or S', and call the angle of the rotation $\phi(t)$ or $-\phi(t)$ (let F' be the quantity to be asked for in each case):

① We observe, in the rotating frame S' with the rotational angle $\phi(t)$, the free motion (of a particle) with respect to the fixed frame S.

①′ We observe, in the rotating frame S' with the rotational angle $-\phi(t)$, the free motion with respect to the fixed frame S.

② We observe, in the fixed frame S', the free motion with respect to the rotating frame S with the rotational angle $\phi(t)$.

②′ We observe, in the fixed frame S', the free motion with respect to the rotating frame S with the rotational angle $-\phi(t)$.
The following commutative diagram holds among these four situations:



(4.4)

Here the vertical arrow represents the interchange of the two words "fixed frame" and "rotating frame." It is obvious here that the diagram is commutative. As discussed at the beginning of this section (i.e., the relativity between the fixed frame and the rotating frame), we have the physically identical result even if we apply the two interchanges, fixed frame ↔ rotating frame and $\phi(t) \leftrightarrow -\phi(t)$. Therefore the two situations connected by a dotted line are physically equivalent:

① = ②′ (equivalent)

and (4.5)

② = ①′ (equivalent)

Therefore we can ask for ①′, instead of asking for ②, corresponding to the interchange $\phi \leftrightarrow -\phi$ in ①, when we know the case ①. This can be verifed using equation (4.3) as in the case of equation (4.2). In the present case the problem is specialized by the condition $F = 0$.

Let us consider the problem of balance of the centrifugal force and Coriolis force with the forces with the reversed directions, respectively. We change the names. We call the fixed frame S and the rotating frame S'. We call hereafter the rotating frame with a positive rotational angle $\phi(t)$ [a negative rotational angle $-\phi(t)$] a p. rot frame (a n. rot frame). We consider the following four situations corresponding further to $F = 0$ or $F' = 0$:

(A) We observe, in a fixed frame S, the free motion ($F' = 0$) with respect to a p. rot frame S'.

(B) We observe, in a p. rot frame S', the free motion ($F = 0$) with respect to a fixed frame S.

(C) We observe, in a n. rot frame S', the free motion ($F = 0$) with respect to a fixed frame S.

(D) We observe, in a fixed frame S, the free motion ($F' = 0$) with respect to a n. rot frame S'.

The following commutative diagram is obtained:

$$\begin{array}{ccc} (A) & \overset{F=0 \leftrightarrow F'=0}{\longleftrightarrow} & (B) \\ \updownarrow & & \updownarrow \phi \leftrightarrow -\phi \\ (D) & \longleftrightarrow & (C) \end{array} \tag{4.6}$$

The commutativity of the diagram is easily verified from equation (4.1). It is to be noted that (A) and (C), or (B) and (D) are no longer physically equivalent here. The relation of these situations is given as follows:

$$\text{(A) [or (B)]} \overset{x \leftrightarrow x',\, F \leftrightarrow F'}{\longleftrightarrow} \text{(C) [or (D)]} \tag{4.7}$$

This fact can easily be understood by noting that we can write as $x' = A \cdot x$ in (A) and (B), and as $x' = A^{-1} \cdot x$ (i.e., $x = A \cdot x'$) in (C) and (D).

For (A) we have, from equation (4.1),

$$A \cdot F = -m \frac{d^2A}{dt^2} A^{-1}x' + 2m\left(\frac{dA}{dt} A^{-1}\right)^2 x' - 2m \frac{dA}{dt} A^{-1} \frac{dx'}{dt} \tag{4.8}$$

which is the representation of the force F in a fixed frame S using the coordinate axes of the S' frame as its base, and which represents the centripetal force and reversed Coriolis force, etc. [see equation (3.5) and following]. On the other hand, for (B) we have also from equation (4.1)

$$F' = m \frac{d^2A}{dt^2} A^{-1}x' - 2m\left(\frac{dA}{dt} A^{-1}\right)^2 x' + 2m \frac{dA}{dt} A^{-1} \frac{dx'}{dt} \tag{4.9}$$

This is nothing but the centrifugal force and Coriolis force, etc., as mentioned in Section 3. Thus each corresponding term in equation (4.8) and equation (4.9) is added to give zero, respectively, and this shows the balance of respective forces. We express this fact simply as

$$A \cdot F|_{F'=0} + F'|_{F=0} = 0 \qquad \text{[(A) and (B); in } S' \text{ frame]} \tag{4.10}$$

Conventionally the centripetal force is introduced using an elastic string or a spring balance as a constraint. However, it is to be noted that we do not need the subsidiary tools here and we can introduce the force of the constraint naturally from the condition of the motion of a particle. Furthermore it is clarified that when we discuss the balance of the fictitious forces we are necessarily implicitly assuming two observers.

Equation (4.10) is the expression of the balance of forces using the rotating coordinate axes (S' frame) as its base. If we use the fixed coordinate axes (S frame) as a base, then we have the expression

$$F|_{F'=0} + A^{-1} \cdot F'|_{F=0} = 0 \qquad \text{[(A) and (B); in } S \text{ frame]} \tag{4.11}$$

We have the expressions for (C) and (D) in the same way:

$$F' = m\frac{d^2(A^{-1})}{dt^2}Ax' - 2m\left(\frac{dA^{-1}}{dt}A\right)^2 x' + 2m\frac{dA^{-1}}{dt}A\frac{dx'}{dt}$$

and

$$A^{-1}\cdot F = -m\frac{d^2(A^{-1})}{dt^2}Ax' + 2m\left(\frac{dA^{-1}}{dt}A\right)^2 x' - 2m\frac{dA^{-1}}{dt}A\frac{dx'}{dt}$$

Therefore the balance of forces can be written as follows:

$$F'|_{F=0} + A^{-1}\cdot F|_{F'=0} = 0 \qquad \text{[(C) and (D); in } S' \text{ frame]} \quad (4.12)$$

and

$$A\cdot F'|_{F=0} + F|_{F'=0} = 0 \qquad \text{[(C) and (D); in } S \text{ frame]} \quad (4.13)$$

Here it should be noted that actually we can derive one equation from another both between equations (4.10) and (4.13), and between equations (4.11) and (4.12), respectively, by the relation (4.7).

5. *The General Time-Dependent Transformation of Motion*

In the previous section we considered especially the rotating frame as an accelerated frame. In this section we consider the general time-dependent transformation of motion,

$$T_3'(t)\colon x \to x' = A(t)\cdot x + a(t) \qquad (5.1)$$

where

$$\det|a_{ij}(t)| = 1 \qquad \text{(for all } t)$$
$${}^tA(t) = A^{-1}(t)$$

Differentiating equation (5.1) we obtain the equations

$$\frac{dx'}{dt} = \frac{dA}{dt}\cdot x + A\cdot\frac{dx}{dt} + \frac{da}{dt} \qquad (5.2)$$

and

$$\frac{d^2x'}{dt^2} = \frac{d^2A}{dt^2}\cdot x + 2\frac{dA}{dt}\cdot\frac{dx}{dt} + A\cdot\frac{d^2x}{dt^2} + \frac{d^2a}{dt^2} \qquad (5.3)$$

If we assume the equation of motion (1.5) in the S frame formally as a vector equation also under the transformation (5.1), we obtain the following transformations according to equation (1.6):

$$\frac{d^2x}{dt^2} \to A(t)\cdot\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2} - \frac{d^2A}{dt^2}\cdot x - 2\frac{dA}{dt}\cdot\frac{dx}{dt} - \frac{d^2a}{dt^2} \qquad (5.4)$$

and

$$F \to A(t) \cdot F \tag{5.5}$$

Here we used equation (5.3). Therefore equation (1.5) is transformed as follows:

$$mA(t) \cdot \frac{d^2x}{dt^2} = m\left\{\frac{d^2x'}{dt^2} - \frac{d^2A}{dt^2}x - 2\frac{dA}{dt}\frac{dx}{dt} - \frac{d^2a}{dt^2}\right\}$$
$$= A(t) \cdot F \tag{5.6}$$

We assume that equation (1.5′) holds in the S' frame, in which F' represents the force acting on the identical particle m in the S' frame. Then we have, from equation (5.6),

$$A(t) \cdot F = F' - m\frac{d^2A}{dt^2}x - 2m\frac{dA}{dt} \cdot \frac{dx}{dt} - m\frac{d^2a}{dt^2} \tag{5.7}$$

On the other hand, from equation (5.1) we have

$$\{T_3'(t)\}^{-1}\colon x' \to x = A^{-1}(t)(x' - a)$$
$$= A^{-1}(t) \cdot x' + a'(t) \tag{5.8}$$

where

$$a'(t) = -A^{-1}(t) \cdot a(t)$$

or

$$a(t) = -A(t) \cdot a'(t) \tag{5.9}$$

From equation (5.8) we obtain

$$\frac{dx}{dt} = A^{-1}\left(\frac{dx'}{dt} - \frac{dA}{dt}A^{-1}x' + A\frac{da'}{dt}\right) \tag{5.10}$$

and

$$\frac{d^2a}{dt^2} = -\frac{d^2A}{dt^2}a' - 2\frac{dA}{dt}\frac{da'}{dt} - A\frac{d^2a'}{dt^2} \tag{5.11}$$

Using these equations we can rewrite equation (5.7) in terms of the quantities in the S' frame, i.e., x' and a', as follows:

$$A \cdot F = F' - m\frac{d^2A}{dt^2}A^{-1}x' + 2m\left(\frac{dA}{dt}A^{-1}\right)^2 x' - 2m\frac{dA}{dt}A^{-1}\frac{dx'}{dt} + mA\frac{d^2a'}{dt^2} \tag{5.12}$$

Comparing this with equation (3.4), we know that only the last term is newly introduced, which is the force arising from coupling of translation and rotation.

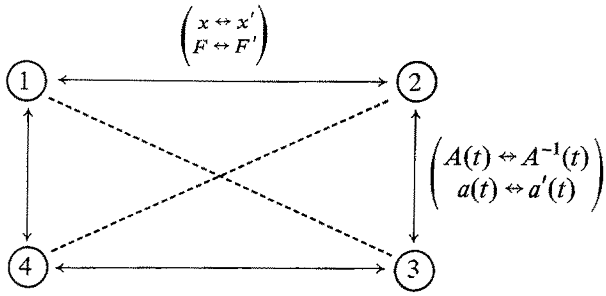
By the relativity between a fixed frame and a moving frame, and by equations (5.1) and (5.8), equation (5.12) must be invariant when we apply the interchanges

$$x \leftrightarrow x', \qquad F \leftrightarrow F' \tag{5.13}$$

and

$$A(t) \leftrightarrow A^{-1}(t), \qquad a(t) \leftrightarrow a'(t) \tag{5.14}$$

In fact we can easily verify the invariance using equations (4.3), (5.8), (5.9), (5.10), and (5.11). Furthermore, we have the commutative diagram



where ①-④ are the following expressions:

① The accelerated frame S' is given by first rotating $[A(t)]$ the fixed frame S, and secondly translating $[a(t)]$ it. [In other words the accelerated frame S' is given by the transformation $T_3'(t)$ to the fixed frame S. Symbolically we express this as $S' = T_3'(t) \cdot S$.]

② The accelerated frame S is given by first rotating $[A(t)]$ the fixed frame S', and secondly translating $[a(t)]$ it; $[S = T_3'(t) \cdot S']$

③ The accelerated frame S is given by first translating $[-a(t)]$ the fixed frame S' (in negative direction), and secondly rotating $[A^{-1}(t)]$ it (in negative direction) $[S = \{T_3'(t)\}^{-1} \cdot S']$.

④ The accelerated frame S' is given by first translating $[-a(t)]$ the fixed frame S (in negative direction), and secondly rotating $[A^{-1}(t)]$ it (in negative direction) $[S' = \{T_3'(t)\}^{-1} \cdot S]$.

In the diagram the horizontal arrow represents the interchange between the S frame and the S' frame, i.e. equation (5.13), and the vertical arrow represents that between the accelerated frame and the fixed frame, i.e., equation (5.14). It is to be noted that because of the noncommutability of translation and rotation, the interchange of two words, the accelerated frame and the fixed frame, in the expression ② gives the expression ③ with S and S' interchanged, in which the order of the two operations rotation and translation is reversed compared to ②. The dotted line represents the physical equivalence.

For the general time-dependent transformation of motion we can also discuss the transformation of the free motion, the balance of the forces, etc. as in the formalism of equations (4.4), (4.6), and (4.10), provided that we displace the word "p. rot frame" (or "n. rot frame") by the word " the frame applied by the (or inverse) transformation $T_3'(t)$" [or $\{T_3'(t)\}^{-1}$], respectively. However, we will not discuss these points farther here.

6. *Discussion and Conclusions*

In the above we applied, though formally, F. Klein's (1849–1925) viewpoint of geometry to Newtonian mechanics [Klein, together with B. Riemann, (1826–1866), is a pioneer of geometry] and tried to point out its importance in mechanics. The group-theoretical viewpoint of geometry will become important for the future development of various branches of dynamics in their foundations. It is true that in the present article we *assumed* formally the vectorial character and the form invariance of Newton's equation. In reality it is known from equation (5.3) that the acceleration d^2x/dt^2 in Newton's equation does not transform as a vector for the general time-dependent transformation of motion. In fact, neither the acceleration nor the velocity dx/dt transforms as a vector, which is shown in equation (5.2). Therefore it is clear that the discussion developed in the present article is hard to accept *if* we are forced to see it only from the purely mathematical viewpoint. How can we improve this point? Let us consider especially the *linear transformation*, i.e., the rotation $A(t)$, among the transformations of motions. Though in the previous sections we treated the rotation $A(t)$ as a *finite* transformation $x \to x'$, there exists the possibility of treating this displacing by the *infinitesimal* transformation

$$dx \to dx'$$

i.e.,

$$dx_j' = \sum_i \frac{\partial x_j'}{\partial x_i} dx_i$$

If we develop this investigation, then we arrive at the conception of a differentiable C^∞ manifold (or still more a real analytic C^ω manifold). On the other hand, if we develop the investigation of an infinitesimal transformation of coordinate *axes*, we arrive at the concept of E. Cartan's Euclidean connection (Yano, 1968), where we can exclude the restriction that the coordinates x_i are the orthogonal Cartesian coordinates. Then we can regard the velocity dx^i/dt as a contravariant vector and the coordinate x^i as itself not a vector, and we can define the acceleration as a tensor by the covariant differential of the velocity dx^i/dt. However, we will give these discussions in succeeding papers, and we give here only the preceding remarks.

In any case it should be noted that we were able to discuss the relativity of forces, the deductive derivation of the fictitious forces and their balance, also for the accelerated frame, i.e., the time-dependent transformation of motion, formally imposing the above two assumptions. Since the times of Galileo and Newton, "the force" is regarded as an absolute quantity (Møller, 1952). It was A. Einstein who, in his theory of general relativity, first tried to introduce the transformation of force under the general transformation of coordinates including accelerated frames. However according to our present investigation it is known that this conception can be formally introduced also in the form of Newton's equation of motion. This fact will have great significance for our new approach to gravitation and motion, which will be presented in succeeding papers.

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